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Löwner equations and dispersionless hierarchies

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Abstract

Using the Hirota representation of dispersionless dKP and dToda hierarchies, we show that the chordal Löwner equations and radial Löwner equations respectively serve as consistency conditions for one-variable reductions of these integrable hierarchies. We also clarify the geometric meaning of this result by relating it to the eigenvalue distribution of normal random matrices in the large N limit.

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1. Introduction

The Löwner equation is a differential equation obeyed by a family of continuously varying univalent conformal maps $G_\lambda(w)$ from the exterior of the unit circle onto a domain with a slit formed by a continuously increasing arc of a fixed curve. The real parameter λ characterizes the length of the arc. Let us normalize the maps so that $G_\lambda(w) = e^\phi w + u_1 + u_2 w^{-1} + \cdots$ as $w \to \infty$ with a real $\phi = \phi(\lambda)$, then the Löwner equation reads

$$\frac{\partial G_\lambda(w)}{\partial \lambda} = -w \frac{\partial G_\lambda(w)}{\partial w} \frac{\sigma(\lambda) + w}{\sigma(\lambda) - w} \frac{\partial \phi(\lambda)}{\partial \lambda}. \tag{1.1}$$

The shape of the fixed curve along which the endpoint of the varying slit runs is encoded by the function $\sigma(\lambda)$. In fact $\sigma(\lambda)$ is the pre-image of the tip of the slit, so it lies on the unit circle: $|\sigma(\lambda)| = 1$. Equation (1.1) was introduced by K Löwner [18] in 1923 in his attempt to prove the Bieberbach conjecture [2, 9, 11]. Now it is referred to as the *radial Löwner equation*.

An analogue of equation (1.1) called *chordal Löwner equation* was introduced in 1999 [13, 25] in other contexts:

$$\frac{\partial H_\lambda(w)}{\partial \lambda} = -\frac{\partial H_\lambda(w)}{\partial w}\frac{1}{U(\lambda)-w}\frac{\partial a_1}{\partial \lambda}. \tag{1.2}$$

Here $H_\lambda(w) = w + a_1 w^{-1} + O(w^{-2})$ as $w \to \infty$ with a_1 being a real coefficient and $U(\lambda)$ is a real-valued function of λ. Under certain conditions, the function $H_\lambda(w)$ conformally maps the upper half plane onto the upper half plane with a slit, with $U(\lambda)$ being the pre-image of the tip.

The Schramm's discovery [17, 25] of the stochastic Löwner evolution (SLE) and its spectacular applications to the conformal field theory (see, e.g., the reviews [1, 4] and references therein) have inspired a renewed interest in the theory of Löwner equations. One of its most interesting aspects is the relation to the integrable hierarchies of nonlinear partial differential equations. This relation was noticed by Gibbons and Tsarev [13] yet before the SLE boom. They have observed that the chordal Löwner equation plays a key role in classifying reductions of the dispersionless KP (dKP) hierarchy [26, 28]. This point was further studied in [20, 35]. Recently, a similar role of the radial Löwner equation in the dispersionless coupled modified KP (dcmKP) hierarchy [30] was elucidated [19, 29]. In the present paper we extend these results to the dispersionless Toda (dToda) hierarchy [27, 28]. Specifically, we characterize a class of reductions of the dToda hierarchy in terms of solutions to the radial Löwner equation. The representation of the hierarchy in the Hirota form appears to be very useful for that purpose. Using the Hirota framework, we also give a more transparent derivation of the chordal Löwner equation as a consistency condition for reductions of the dKP hierarchy.

In short, our main result is the following[4]. Given any solution $G_\lambda(w)$ to the radial Löwner equation (1.1) with any (continuous) $\sigma(\lambda)$, the Lax function $\mathcal{L}(w;\boldsymbol{t}) = G_\lambda(w)$ solves the Lax equations of the dToda hierarchy provided the dependence $\lambda = \lambda(\boldsymbol{t})$ is given by the system of hydrodynamic-type

$$\frac{\partial \lambda}{\partial t_n} = \xi_n(\lambda)\frac{\partial \lambda}{\partial t_0}. \tag{1.3}$$

In these equations, the functions $\xi_n(\lambda)$ are constructed in a canonical way from $\mathcal{L}(w;\boldsymbol{t})$ and $\sigma(\lambda)$. In the context of the dKP and dcmKP hierarchies, this result was earlier established in [13, 19, 20, 29]. Here we also prove the converse. Let $\mathcal{L}(w;\boldsymbol{t})$ be the Lax function of the dToda hierarchy. Suppose one has a *one-variable reduction* of the hierarchy, i.e., the Lax function depends on the hierarchical times $\boldsymbol{t} = (\ldots, t_{-1}, t_0, t_1, \ldots)$ via a single function $\lambda(\boldsymbol{t})$:

$$\mathcal{L}(w;\boldsymbol{t}) = \mathcal{L}(w, \lambda(\boldsymbol{t})).$$

Then consistency of this ansatz with the hierarchy implies that the function of two variables $\mathcal{L}(w,\lambda)$ obeys the radial Löwner equation (1.1) with some $\sigma(\lambda)$ while the time dependence of λ is determined from system (1.3). A similar statement holds true for the dKP hierarchy with the chordal Löwner equation instead of the radial one.

The geometric meaning of this result and, more generally, that of reductions of the dToda hierarchy, is clarified in section 6. It is known that solutions of the dToda hierarchy obeying certain reality conditions generate univalent conformal maps of planar domains. The domain to be mapped (onto some fixed reference domain) depends on which solution of the hierarchy one takes as well as on values of the hierarchical 'times' t_n. Generic solutions correspond to mappings of planar domains bounded by smooth simple curves, with the times being suitably

[4] To make this introduction as short as possible, we do not discuss the second Lax function of the dToda hierarchy, for which similar results hold true. For precise statements, see propositions 5.6 and 5.11.

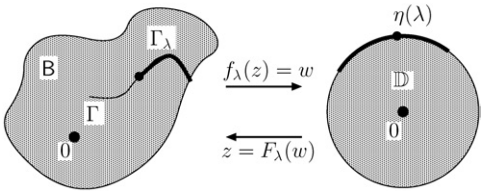

Figure 1. Conformal maps F_λ, f_λ.

defined moments of the domain [15, 21, 23, 34, 36]. Non-generic solutions, or *reductions* of the hierarchy, are known to be related to conformal maps of domains with highly singular boundaries, such as slit domains [13, 35].

Both types of solutions and conformal maps corresponding to them can be understood in terms of the model of normal random matrices [6, 7] in the large N limit. Actually, we use only the eigenvalue integral representation, which is the partition function of the 2D Coulomb gas in external field. The matrix models are known to be solvable (even at finite N) in terms of integrable hierarchies, which become dispersionless as $N \to \infty$. At the same time, the domains subject to conformal maps under study appear in this context as complements of the supports of eigenvalues in the $N \to \infty$ matrix model. This picture provides a transparent geometric meaning of the reductions and conformal maps of slit domains corresponding to them.

2. Löwner equations

2.1. Radial Löwner equation

The Löwner theory was introduced by Löwner in [18]. It plays an important role in solving the Bieberbach conjecture [2, 9, 33]. Here we briefly recall the necessary ingredients in the context we need. One can refer to [8, 11, 24] for a complete exploration of the theory.

Let $\mathbb{D}$ be the unit disc and $\mathbb{D}^*$ its exterior. Let B be a simply connected domain in the complex plane $\mathbb{C}$ containing the origin and let $\Gamma : [0, \infty) \to \mathbb{C}$ be a Jordan arc growing inside B, i.e., $\Gamma(0)$ is a point lying on the boundary of B, and for $\mu \in (0, \infty)$, $\Gamma(\mu) \in \mathsf{B}$. Let $\Gamma_\lambda = \Gamma|_{[0,\lambda]}$ be the arc of Γ between 0 and λ and let $\mathsf{B}_\lambda = \mathsf{B}\backslash\Gamma_\lambda$ be the corresponding slit domain. By the Riemann mapping theorem, there exists a unique conformal map F_λ mapping the unit disc $\mathbb{D}$ onto the domain B_λ and satisfying $F_\lambda(0) = 0$, $F_\lambda'(0) = \mathrm{e}^{-\phi(\lambda)} > 0$. Let $\eta(\lambda)$ be the point on $S^1 = \partial\mathbb{D}$ mapped to the tip of Γ_λ by F_λ, i.e., $F_\lambda(\eta(\lambda)) = \Gamma(\lambda)$. Then $F(w, \lambda) = F_\lambda(w)$ satisfies the following differential equation:

$$\frac{\partial F(w,\lambda)}{\partial\lambda} = -w\frac{\partial F(w,\lambda)}{\partial w}\frac{\eta(\lambda)+w}{\eta(\lambda)-w}\frac{\partial\phi(\lambda)}{\partial\lambda}, \qquad w \in \mathbb{D}, \lambda \in [0,\infty), \tag{2.1}$$

called the (radial) Löwner equation. Let f_λ be the inverse function of F_λ. The corresponding Löwner equation for $f(z, \lambda) = f_\lambda(z)$ is

$$\frac{\partial f(z,\lambda)}{\partial\lambda} = f(z,\lambda)\frac{\eta(\lambda)+f(z,\lambda)}{\eta(\lambda)-f(z,\lambda)}\frac{\partial\phi(\lambda)}{\partial\lambda}, \qquad z \in \mathsf{B}_\lambda, \lambda \in [0,\infty).$$

(See figure 1.)

Conversely, given a measurable function $\eta : [0, \infty) \to S^1$ and an increasing differentiable function $\phi : [0, \infty) \to \mathbb{R}$, the Löwner equation (2.1) with the initial condition $F_0(\mathbb{D}) = \mathsf{B}$,

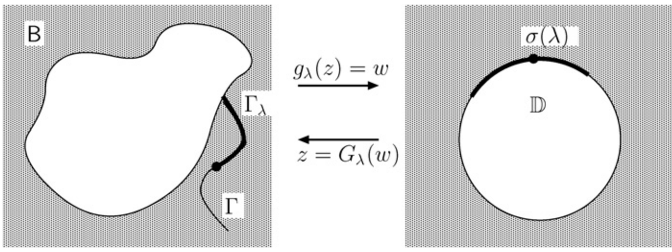

Figure 2. Conformal maps G_λ, g_λ.

$F_0(0) = 0$, $F_0'(0) = \mathrm{e}^{-\phi(0)}$ always has a unique solution $F_\lambda : \mathbb{D} \to \mathbb{C}$ which maps the unit disc to a family of continuously shrinking domains $F_\lambda(\mathbb{D})$.[5]

In case when B is a simply connected domain in $\hat{\mathbb{C}}$ that contains ∞ and $\Gamma : [0, \lambda) \to \hat{\mathbb{C}}$ is a growing Jordan arc in B, let $\Gamma_\lambda = \Gamma|_{[0,\lambda]}$, and let G_λ be the unique conformal map from the exterior disc $\mathbb{D}^*$ onto $\mathsf{B}_\lambda = \mathsf{B}\backslash\Gamma_\lambda$ normalized such that $G_\lambda(\infty) = \infty$, $G_\lambda'(\infty) = \mathrm{e}^{\phi(\lambda)} > 0$ and let g_λ be the inverse function of G_λ. Then $G(w, \lambda) = G_\lambda(w)$ and $g(z, \lambda) = g_\lambda(z)$ satisfy the Löwner equations

$$\begin{aligned} \frac{\partial G(w,\lambda)}{\partial\lambda} &= -w\frac{\partial G(w,\lambda)}{\partial w}\frac{\sigma(\lambda)+w}{\sigma(\lambda)-w}\frac{\partial\phi(\lambda)}{\partial\lambda}, && w \in \mathbb{D}^*, \lambda \in [0,\infty), \\ \frac{\partial g(z,\lambda)}{\partial\lambda} &= g(z,\lambda)\frac{\sigma(\lambda)+g(z,\lambda)}{\sigma(\lambda)-g(z,\lambda)}\frac{\partial\phi(\lambda)}{\partial\lambda}, && z \in \mathsf{B}_\lambda, \lambda \in [0,\infty). \end{aligned} \tag{2.2}$$

(See figure 2) Here $\sigma(\lambda)$ is the point on S^1 mapped to the tip of Γ_λ by G_λ.

In the special case when the domain B is the exterior disc $\mathbb{D}^*$, let $J_\lambda \subset S^1$ be the pre-image of Γ_λ under the map G_λ. Then G_λ maps the arc $S^1\backslash J_\lambda$ to an arc of S^1. By the Schwarz reflection principle, the map G_λ has an analytic continuation $\tilde{G}_\lambda$ defined on $\mathbb{D}\cup\mathbb{D}^*\cup(S^1\backslash J_\lambda)$, with the image $\hat{\mathbb{C}}\backslash(\Gamma_\lambda \cup \tilde{\Gamma}_\lambda)$, where $\tilde{\Gamma}_\lambda$ is the mirror image of Γ_λ under reflection in the unit circle, i.e. $\tilde{\Gamma}_\lambda = \{1/\bar{z} : z \in \Gamma_\lambda\}$. The map $F_\lambda = \tilde{G}_\lambda|_{\mathbb{D}}$ is given explicitly by

$$F_\lambda(w) = \overline{G_\lambda(1/\bar{w})}^{-1}.$$

Using this relation, it is easy to check that $F(w, \lambda) = F_\lambda(w)$ satisfies the Löwner equation (2.1) with $\eta(\lambda) = \sigma(\lambda)$. Therefore, the differential equations for $G(w, \lambda)$ and $F(w, \lambda)$ are the same but defined in different domains.

2.2. *Chordal Löwner Equation*

The chordal Löwner equation [13, 17, 25] is an analogue of the radial Löwner equation for conformal maps on the upper half plane $\mathbb{H}$. We say that a conformal map $H : \mathbb{H} \to \mathbb{H}$ that has continuous extension to $\partial\mathbb{H}$ is normalized with respect to the point ∞ if it has a Laurent series expansion of the form

$$H(w) = w + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots.$$

Let $H_\lambda : \mathbb{H} \to \mathbb{H}$ be a sequence of normalized conformal maps such that $H_\lambda(\mathbb{H}) = \mathbb{H}\backslash\Gamma_\lambda$, where $\Gamma : [0, \infty) \to \mathbb{C}$ is a growing Jordan arc in $\mathbb{H}$ and $\Gamma_\lambda = \Gamma|_{[0,\lambda]}$. Denote by $U(\lambda)$ the

[5] However, $F_\lambda(\mathbb{D})$ are not necessarily slit domains, see [16, 22].

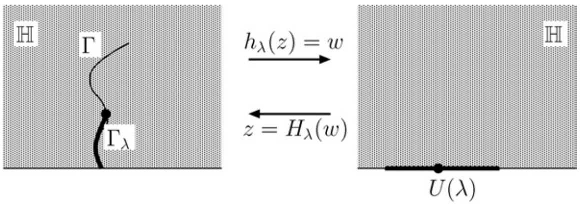

Figure 3. Conformal maps H_λ, h_λ.

point on $\mathbb{R}$ that is mapped to the tip of Γ_λ by H_λ and let h_λ be the inverse function of H_λ. Then $H(w,\lambda) = H_\lambda(w)$ and $h(z,\lambda) = h_\lambda(z)$ satisfy the differential equations:

$$\begin{aligned} \frac{\partial H(w,\lambda)}{\partial\lambda} &= -\frac{\partial H(w,\lambda)}{\partial w}\frac{1}{U(\lambda)-w}\frac{\partial a_1(\lambda)}{\partial\lambda}, && \lambda\in[0,\infty), \\ \frac{\partial h(z,\lambda)}{\partial\lambda} &= \frac{1}{U(\lambda)-h(z,\lambda)}\frac{\partial a_1(\lambda)}{\partial\lambda}, && \lambda\in[0,\infty), \end{aligned} \tag{2.3}$$

which are called the chordal Löwner equations for H_λ and h_λ, respectively. (See figure 3.)

3. Grunsky coefficients and Faber polynomials

In this section we review the definitions of Faber polynomials and Grunsky coefficients. For details, see [11, 24, 31]. Let

$$f(z) = rz + \alpha_2 z^2 + \alpha_3 z^3 + \cdots \qquad \text{and} \qquad g(z) = r^{-1}z + \beta_0 + \frac{\beta_1}{z} + \frac{\beta_2}{z^2} + \cdots$$

be functions univalent in a neighbourhood of 0 and ∞, respectively. The Grunsky coefficients $b_{m,n}$, $m,n\in\mathbb{Z}$, of the pair (f,g) are defined by the expansions

$$\begin{aligned} \log\frac{g(z_1)-g(z_2)}{z_1-z_2} &= -\log r - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{m,n} z_1^{-m} z_2^{-n} \\ \log\frac{g(z_1)-f(z_2)}{z_1} &= -\log r - \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{m,-n} z_1^{-m} z_2^{n} \\ \log\frac{f(z_1)-f(z_2)}{z_1-z_2} &= -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{-m,-n} z_1^{m} z_2^{n} \end{aligned} \tag{3.1}$$

and $b_{-m,n} = b_{n,-m}$ for $n \geqslant 1, m \geqslant 0$. By definition, $b_{m,n} = b_{n,m}$ for all $n, m \in \mathbb{Z}$, $b_{0,0} = -\log r$ and

$$\log\frac{g(z)}{z} = -\log r - \sum_{n=1}^{\infty} b_{n,0} z^{-n}, \qquad \log\frac{f(z)}{z} = \log r - \sum_{n=1}^{\infty} b_{-n,0} z^{n}. \tag{3.2}$$

Using these formulae, the second and third equations in (3.1) can be rewritten as

$$\begin{aligned} \log\left(1-\frac{f(z_2)}{g(z_1)}\right) &= -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{m,-n} z_1^{-m} z_2^{n} \\ \log\frac{f(z_1)^{-1}-f(z_2)^{-1}}{z_1^{-1}-z_2^{-1}} &= -\log r - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{-m,-n} z_1^{m} z_2^{n}. \end{aligned} \tag{3.3}$$

For g alone, $b_{m,n}$, $m, n \geqslant 0$ are called the Grunsky coefficients of g.

The Faber polynomials $\Phi_n(w), n \geqslant 1$ of g and Faber polynomials $\Psi_n, n \geqslant 1$ of f are defined by

$$\log \frac{g(z) - w}{r^{-1} z} = -\sum_{n=1}^{\infty} \frac{\Phi_n(w)}{n} z^{-n}, \qquad \log \frac{w - f(z)}{w} = \log \frac{f(z)}{rz} - \sum_{n=1}^{\infty} \frac{\Psi_n(w)}{n} z^n, \quad (3.4)$$

so that

$$\Phi_n(w) = (G(w)^n)_{\geqslant 0}, \qquad \Psi_n(w) = (F(w)^{-n})_{\leqslant 0},$$

where F and G are the inverse functions of f and g, respectively, and for $S \subset \mathbb{Z}$ we denote $\left(\sum_{k\in\mathbb{Z}} A_k w^k\right)_S = \sum_{k\in S} A_k w^k$. Taking derivatives with respect to $\log w$, we have

$$\frac{w}{w - g(z)} = -\sum_{n=1}^{\infty} \frac{w\Phi_n'(w)}{n} z^{-n}, \qquad \frac{f(z)}{w - f(z)} = -\sum_{n=1}^{\infty} \frac{w\Psi_n'(w)}{n} z^n. \quad (3.5)$$

4. Dispersionless hierarchies

We briefly review the dispersionless Toda (dToda) hierarchy, the dispersionless KP (dKP) hierarchy and their tau functions. For details, see [26–28].

4.1. Dispersionless KP hierarchy

The fundamental quantity in the dKP hierarchy is a formal power series

$$\mathcal{L}(w; \boldsymbol{t}) = w + \sum_{n=1}^{\infty} u_{n+1}(\boldsymbol{t}) w^{-n} \quad (4.1)$$

with coefficients depending on the independent variables $\boldsymbol{t} = (x + t_1, t_2, t_3, \ldots)$. The Lax representation of the dKP hierarchy is

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \mathcal{B}_n = (\mathcal{L}^n)_{\geqslant 0}. \quad (4.2)$$

Here $\{\cdot, \cdot\}$ is the Poisson bracket

$$\{h_1, h_2\} = \frac{\partial h_1}{\partial w}\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial w}.$$

Let $k(z; \boldsymbol{t})$ be the inverse of $\mathcal{L}(w; \boldsymbol{t})$, i.e. $k(\mathcal{L}(w; \boldsymbol{t}); \boldsymbol{t}) = w$ and $\mathcal{L}(k(z; \boldsymbol{t}); \boldsymbol{t}) = z$. There exists a tau function $\tau_{\mathrm{dKP}}(\boldsymbol{t})$ that generates the Grunsky coefficients $b_{m,n}(\boldsymbol{t}), m, n \geqslant 1$ of $k(z; \boldsymbol{t})$. More precisely,

$$\frac{\partial^2 \log \tau_{\mathrm{dKP}}(\boldsymbol{t})}{\partial t_m \partial t_n} = -mnb_{m,n}(\boldsymbol{t}). \quad (4.3)$$

Introduce the operator

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n},$$

and set $\mathcal{F} = \log \tau_{\mathrm{dKP}}$ (the 'free energy'). Then the first equation of (3.1) can be written as

$$\log \frac{k(z_1) - k(z_2)}{z_1 - z_2} = D(z_1)D(z_2)\mathcal{F}, \qquad k(z) = z - D(z)\partial_{t_1}\mathcal{F}, \quad (4.4)$$

which is the Hirota equation for the dKP hierarchy [5, 10, 12, 28] . Conversely, it was proved [3, 5, 12, 31] that the Hirota equation implies the dKP hierarchy. We reformulate this statement below in the form we need later.

Proposition 4.1. *Let $k(z;\boldsymbol{t})$ be the inverse of the formal power series $\mathcal{L}(w;\boldsymbol{t})$ of the form (4.1) and let $b_{m,n}(\boldsymbol{t})$ be the Grunsky coefficients of $k(z;\boldsymbol{t})$. If there exists a function $\tau_{\mathrm{dKP}}(\boldsymbol{t})$ satisfying (4.3), then $\mathcal{L}(w;\boldsymbol{t})$ is a solution of the dKP hierarchy (4.2).*

4.2. Dispersionless Toda hierarchy

The fundamental quantities in the dToda hierarchy are two formal power series in w :

$$\mathcal{L}(w;\boldsymbol{t}) = r(\boldsymbol{t})w + \sum_{n=0}^{\infty} u_{n+1}(\boldsymbol{t})w^{-n}, \qquad \tilde{\mathcal{L}}^{-1}(w;\boldsymbol{t}) = r(\boldsymbol{t})w^{-1} + \sum_{n=0}^{\infty} \tilde{u}_{n+1}(\boldsymbol{t})w^{n}. \tag{4.5}$$

Here $u_n(\boldsymbol{t})$ and $\tilde{u}_n(\boldsymbol{t})$ are functions of the independent variables $t_n, n \in \mathbb{Z}$, which we denote collectively by $\boldsymbol{t}$. The Lax representation is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{L}\}_T, & \frac{\partial \mathcal{L}}{\partial t_{-n}} &= \{\tilde{\mathcal{B}}_n, \mathcal{L}\}_T, \\ \frac{\partial \tilde{\mathcal{L}}}{\partial t_n} &= \{\mathcal{B}_n, \tilde{\mathcal{L}}\}_T, & \frac{\partial \tilde{\mathcal{L}}}{\partial t_{-n}} &= \{\tilde{\mathcal{B}}_n, \tilde{\mathcal{L}}\}_T. \end{aligned} \tag{4.6}$$

Here

$$\mathcal{B}_n = (\mathcal{L}^n)_{>0} + \tfrac{1}{2}(\mathcal{L}^n)_0, \qquad \tilde{\mathcal{B}}_n = (\tilde{\mathcal{L}}^{-n})_{<0} + \tfrac{1}{2}(\tilde{\mathcal{L}}^{-n})_0,$$

and $\{\cdot,\cdot\}_T$ is the Poisson bracket for the dToda hierarchy

$$\{h_1, h_2\}_T = w\frac{\partial h_1}{\partial w}\frac{\partial h_2}{\partial t_0} - w\frac{\partial h_1}{\partial t_0}\frac{\partial h_2}{\partial w}.$$

Let $p(z;\boldsymbol{t})$ and $\tilde{p}(z;\boldsymbol{t})$ be the inverses of $\mathcal{L}(w;\boldsymbol{t})$ and $\tilde{\mathcal{L}}(w;\boldsymbol{t})$, respectively, i.e. $p(\mathcal{L}(w;\boldsymbol{t});\boldsymbol{t}) = w$, $\mathcal{L}(p(z;\boldsymbol{t});\boldsymbol{t}) = z$, $\tilde{p}(\tilde{\mathcal{L}}(w;\boldsymbol{t});\boldsymbol{t}) = w$ and $\tilde{\mathcal{L}}(\tilde{p}(z;\boldsymbol{t});\boldsymbol{t}) = z$. There exists a tau function $\tau_{\mathrm{dToda}}(\boldsymbol{t})$ which generates the Grunsky coefficients $b_{m,n}(\boldsymbol{t}), m, n \in \mathbb{Z}$ of $(\tilde{p}(z;\boldsymbol{t}), p(z;\boldsymbol{t}))$. More precisely,

$$\frac{\partial^2 \log \tau_{\mathrm{dToda}}(\boldsymbol{t})}{\partial t_m \partial t_n} = \begin{cases} -|mn| b_{mn}(\boldsymbol{t}), & \text{if} \quad m \neq 0, n \neq 0 \\ |m| b_{m,0}(\boldsymbol{t}), & \text{if} \quad m \neq 0, n = 0 \\ -2b_{00}(\boldsymbol{t}), & \text{if} \quad m = n = 0. \end{cases} \tag{4.7}$$

Let us introduce the operators

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\frac{\partial}{\partial t_n}, \qquad \tilde{D}(z) = \sum_{n=1}^{\infty} \frac{z^{n}}{n}\frac{\partial}{\partial t_{-n}} \tag{4.8}$$

and define the free energy $\mathcal{F}$ by $\mathcal{F} = \log \tau_{\mathrm{dToda}}$. Then we can rewrite the first equation in (3.1), equations (3.3) and (3.2) in the form

$$\begin{aligned} &\log \frac{p(z_1) - p(z_2)}{z_1 - z_2} = -\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial t_0^2} + D(z_1)D(z_2)\mathcal{F} \\ &\log\left(1 - \frac{\tilde{p}(z_2)}{p(z_1)}\right) = D(z_1)\tilde{D}(z_2)\mathcal{F} \\ &\log \frac{\tilde{p}(z_1)^{-1} - \tilde{p}(z_2)^{-1}}{z_1^{-1} - z_2^{-1}} = -\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial t_0^2} + \tilde{D}(z_1)\tilde{D}(z_2)\mathcal{F} \\ &\log \frac{p(z)}{z} = -\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial t_0^2} - D(z)\partial_{t_0}\mathcal{F}, \qquad \log \frac{\tilde{p}(z)}{z} = \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial t_0^2} - \tilde{D}(z)\partial_{t_0}\mathcal{F}. \end{aligned} \tag{4.9}$$

This is the system of Hirota equations for the dToda hierarchy [3, 21, 23, 34, 36]. Conversely, it was shown in [3, 5, 12, 31] that the Hirota equations imply the dToda hierarchy. We reformulate this statement below in the form we need.

Proposition 4.2. *Let $p(z; \boldsymbol{t})$ and $\tilde{p}(z; \boldsymbol{t})$ be the inverses of the formal power series $\mathcal{L}(w; \boldsymbol{t})$ and $\tilde{\mathcal{L}}(w; \boldsymbol{t})$ of the form (4.5) and let $b_{m,n}(\boldsymbol{t})$ be the Grunsky coefficients of $(\tilde{p}(z; \boldsymbol{t}), p(z; \boldsymbol{t}))$. If there exists a function $\tau_{\mathrm{dToda}}(\boldsymbol{t})$ satisfying (4.7), then $(\mathcal{L}(w; \boldsymbol{t}), \tilde{\mathcal{L}}(w; \boldsymbol{t}))$ is a solution of the dToda hierarchy (4.6).*

5. One-variable reductions of dispersionless hierarchies

In this section, we consider the one-variable reductions of the dKP and dToda hierarchies from the perspective of their Hirota equations.

5.1. One-variable reduction of dKP hierarchy

Proposition 5.1. *Suppose that $\mathcal{L}(w; \boldsymbol{t})$ is a solution of the dKP hierarchy whose dependence on $\boldsymbol{t} = (x + t_1, t_2, t_3, \ldots)$ is through a single variable λ. Namely, there exists a function $\lambda(\boldsymbol{t})$ of $\boldsymbol{t}$ such that*

$$\mathcal{L}(w; \boldsymbol{t}) = \mathcal{L}(w, \lambda(\boldsymbol{t})) = w + \sum_{n=1}^{\infty} u_{n+1}(\lambda(\boldsymbol{t}))w^{-n}.$$

Then $\mathcal{L}(w, \lambda)$ satisfies the chordal Löwner equation (2.3) with respect to λ. Moreover, let $k(z, \lambda)$ be the inverse function of $\mathcal{L}(w, \lambda)$, $\Phi_n(w, \lambda)$, $n \geqslant 1$, be the Faber polynomials of $k(z, \lambda)$ and $\chi_n(\lambda) = \Phi_n'(U(\lambda), \lambda)$, then $\lambda(\boldsymbol{t})$ satisfies the hydrodynamic-type equation

$$\frac{\partial \lambda}{\partial t_n} = \chi_n(\lambda) \frac{\partial \lambda}{\partial t_1}. \tag{5.1}$$

Proof. Let $\mathcal{F}(\boldsymbol{t}) = \log \tau_{\mathrm{dKP}}(\boldsymbol{t})$ be the free energy of the solution $\mathcal{L}(w, \lambda(\boldsymbol{t}))$. From the second equation in (4.4) we have

$$D(z_1)k(z_2, \lambda) = D(z_2)k(z_1, \lambda).$$

Setting $z_1 = z$ and comparing the coefficient of z_2^{-1} on both sides, we obtain

$$D(z)u = -\partial_{t_1} k(z, \lambda) = -\partial_\lambda k(z, \lambda)\partial_{t_1}\lambda, \qquad \text{where} \quad u = \frac{\partial^2 \mathcal{F}}{\partial t_1^2} = u_2(\lambda(\boldsymbol{t})).$$

Suppose $\partial_\lambda u \neq 0$, then we have by the chain rule:

$$D(z)\lambda = -\partial_\lambda k(z, \lambda) \frac{\partial_{t_1}\lambda}{\partial_\lambda u} \tag{5.2}$$

and

$$D(z_2)k(z_1, \lambda) = -\partial_\lambda k(z_1, \lambda)\partial_\lambda k(z_2, \lambda) \frac{\partial_{t_1}\lambda}{\partial_\lambda u}.$$

Applying ∂_{t_1} to both sides of the first equation in (4.4), we get

$$\frac{\partial_\lambda k(z_1, \lambda) - \partial_\lambda k(z_2, \lambda)}{k(z_1, \lambda) - k(z_2, \lambda)} \partial_{t_1}\lambda = D(z_2)(z_1 - k(z_1, \lambda)) = \partial_\lambda k(z_1, \lambda)\partial_\lambda k(z_2, \lambda) \frac{\partial_{t_1}\lambda}{\partial_\lambda u}.$$

If $\partial_{t_1}\lambda \neq 0$, then

$$(\partial_\lambda k(z_1, \lambda) - \partial_\lambda k(z_2, \lambda))\, \partial_\lambda u = \partial_\lambda k(z_1, \lambda)\partial_\lambda k(z_2, \lambda)(k(z_1, \lambda) - k(z_2, \lambda)),$$

which means that

$$(\partial_\lambda k(z,\lambda))^{-1}\partial_\lambda u + k(z,\lambda) = U(\lambda) \tag{5.3}$$

is independent of z. This gives

$$\frac{\partial k(z,\lambda)}{\partial\lambda} = \frac{1}{U(\lambda)-k(z,\lambda)}\frac{\partial u}{\partial\lambda},$$

which is the chordal Löwner equation (2.3) for $k(z,\lambda)$.

Now, from equation (5.2) and the first equation in (3.5), we have

$$\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial\lambda}{\partial t_n} = \frac{1}{k(z,\lambda)-U(\lambda)}\frac{\partial\lambda}{\partial t_1} = \sum_{n=1}^{\infty}\frac{z^{-n}}{n}\Phi_n'(U(\lambda),\lambda)\frac{\partial\lambda}{\partial t_1}.$$

Therefore, with $\chi_n(\lambda) = \Phi_n'(U(\lambda),\lambda)$, we have

$$\frac{\partial\lambda}{\partial t_n} = \chi_n(\lambda)\frac{\partial\lambda}{\partial t_1}.$$

□

Remark 5.2. If we assume that the function $\mathcal{L}(w;\boldsymbol{t})$ maps the upper half plane $\mathbb{H}$ onto a slit sub-domain of the upper half plane, then by letting $z\to\partial\mathbb{H}$ in (5.3), we find that $U(\lambda)\in\mathbb{R}$.

Proposition 5.1 implies that for $\mathcal{L}(w,\lambda(\boldsymbol{t}))$ to satisfy the dKP hierarchy, $\lambda(\boldsymbol{t})$ must satisfy the hydrodynamic-type equation (5.1). In fact, (5.1) can be solved by the general hodograph method of Tsarev [32]:

Lemma 5.3. *Let $R(\lambda)$ be any function of λ. If $\lambda(\boldsymbol{t})$ is defined implicitly by the hodograph relation*

$$x + t_1 + \sum_{n=2}^{\infty}\chi_n(\lambda)t_n = R(\lambda), \tag{5.4}$$

then $\lambda(\boldsymbol{t})$ satisfies (5.1).

Proof. A straightforward computation. □

Remark 5.4. If all the coefficients of the series $\mathcal{L}(w,\lambda)$ are real, then $\Phi_n(w,\lambda)$ is a polynomial in w with real coefficients. Therefore, imposing the condition that all the variables t_n are real, the relation (5.4) can be solved for λ as a real-valued function in a certain domain of $\boldsymbol{t}$.

As a converse to proposition 5.1, Gibbons and Tsarev [35], Yu and Gibbons [13], Mañas, Martínez Alonso and Medina [20] and others have shown that a solution of the chordal Löwner equation (2.3) together with equation (5.1) gives rise to a solution of the dKP hierarchy. Here we give an independent proof using the Hirota equations.

Proposition 5.5. *Let $H(w,\lambda)$ be a solution of the chordal Löwner equation (2.3) and $\lambda(\boldsymbol{t})$ a solution of equation (5.1). Then $\mathcal{L}(w;\boldsymbol{t}) = H(w,\lambda(\boldsymbol{t}))$ is a solution of the dKP hierarchy (4.2).*

Proof. Let $h(z,\lambda)$ be the inverse function of $H(w,\lambda)$ and let $b_{m,n}(\boldsymbol{t}) = b_{m,n}(\lambda(\boldsymbol{t}))$, $m,n\geqslant 1$ be the Grunsky coefficients of $h(z,\lambda)$. Applying ∂_{t_k} to both sides of the first equation of (3.1) and using the Löwner equation for $h(z,\lambda)$ (2.3), equation (5.1) and the first equation in (3.5), we have

$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\partial b_{m,n}(\boldsymbol{t})}{\partial t_k}z_1^{-m}z_2^{-n} = \frac{\partial_\lambda h(z_1,\lambda)-\partial_\lambda h(z_2,\lambda)}{h(z_1,\lambda)-h(z_2,\lambda)}\frac{\partial\lambda}{\partial t_k}$$
$$= \frac{1}{(U(\lambda)-h(z_1))(U(\lambda)-h(z_2))}\frac{\partial a_1(\lambda)}{\partial\lambda}\chi_k(\lambda)\frac{\partial\lambda}{\partial t_1}$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\chi_m(\lambda)\chi_n(\lambda)z_1^{-m}z_2^{-n}\frac{\partial a_1(\lambda)}{\partial\lambda}\chi_k(\lambda)\frac{\partial\lambda}{\partial t_1}.$$

Comparing coefficients gives

$$-mn\frac{\partial b_{m,n}(\boldsymbol{t})}{\partial t_k} = \chi_m(\lambda)\chi_n(\lambda)\chi_k(\lambda)\frac{\partial a_1(\lambda)}{\partial\lambda}\frac{\partial\lambda}{\partial t_1},$$

which is completely symmetric in m, n and k. Therefore, there exists a function $\mathcal{F}(\boldsymbol{t})$ such that

$$\frac{\partial^2\mathcal{F}(\boldsymbol{t})}{\partial t_m\partial t_n} = -mnb_{m,n}(\boldsymbol{t}).$$

By proposition 4.1, the conclusion follows. □

5.2. One-variable reduction of dToda hierarchy

Proposition 5.6. *Suppose that $(\mathcal{L}(w;\boldsymbol{t}),\tilde{\mathcal{L}}(w;\boldsymbol{t}))$ is a solution of the dToda hierarchy whose dependence on $\boldsymbol{t}=(\ldots,t_{-2},t_{-1},t_0,t_1,t_2,\ldots)$ is through a single variable λ. Namely, there exists a function $\lambda(\boldsymbol{t})$ of $\boldsymbol{t}$ such that*

$$\mathcal{L}(w;\boldsymbol{t}) = \mathcal{L}(w,\lambda(\boldsymbol{t})) = r(\lambda(\boldsymbol{t}))w + \sum_{n=0}^{\infty}u_{n+1}(\lambda(\boldsymbol{t}))w^{-n}$$
$$\tilde{\mathcal{L}}(w;\boldsymbol{t})^{-1} = \tilde{\mathcal{L}}(w,\lambda(\boldsymbol{t}))^{-1} = r(\lambda(\boldsymbol{t}))w^{-1} + \sum_{n=0}^{\infty}\tilde{u}_{n+1}(\lambda(\boldsymbol{t}))w^{n}.$$

Then $\mathcal{L}(w,\lambda)$ and $\tilde{\mathcal{L}}(w,\lambda)$ satisfy the radial Löwner equations (2.2) and (2.1), respectively, in which $\phi(\lambda)=\log r(\lambda)$ and $\eta(\lambda)=\sigma(\lambda)$. Moreover, if $p(z,\lambda)$, $\tilde{p}(z,\lambda)$ are the inverse functions of $\mathcal{L}(w,\lambda)$, $\tilde{\mathcal{L}}(w,\lambda)$, respectively; $\Phi_n(w,\lambda)$, $\Psi_n(w,\lambda)$, $n\geqslant 1$ are Faber polynomials of $p(z,\lambda)$ and $\tilde{p}(z,\lambda)$, respectively; $\xi_n(\lambda)$, $n\in\mathbb{Z}$ are defined by

$$\xi_n(\lambda) = \begin{cases}\sigma(\lambda)\Phi_n'(\sigma(\lambda),\lambda), & \text{if}\quad n\geqslant 1\\ 1, & \text{if}\quad n=0\\ \sigma(\lambda)\Psi_n'(\sigma(\lambda),\lambda), & \text{if}\quad n\leqslant -1,\end{cases}$$

then $\lambda(\boldsymbol{t})$ satisfies the hydrodynamic-type equation

$$\frac{\partial\lambda}{\partial t_n} = \xi_n(\lambda)\frac{\partial\lambda}{\partial t_0}. \tag{5.5}$$

Proof. Let $\mathcal{F}(\boldsymbol{t})=\log\tau_{\mathrm{dToda}}(\boldsymbol{t})$ be the free energy of the solution. From the fourth equation in (4.9), we have

$$\left(\tfrac{1}{2}\partial_{t_0}+D(z_1)\right)\log p(z_2,\lambda) = \left(\tfrac{1}{2}\partial_{t_0}+D(z_2)\right)\log p(z_1,\lambda),$$

Setting $z_1 = z$ and tending $z_2 \to \infty$, we have

$$D(z)\phi = -\frac{1}{2}\partial_{t_0}\log\left(rp(z,\lambda)\right), \qquad \text{where} \quad \phi = \log r = \frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial t_0^2}.$$

Suppose $\partial_\lambda\phi \neq 0$, then we have by the chain rule:

$$D(z)\lambda = -\frac{1}{2}\partial_\lambda \log\left(rp(z,\lambda)\right)\frac{\partial_{t_0}\lambda}{\partial_\lambda\phi}. \tag{5.6}$$

Applying ∂_{t_0} to both sides of the first equation in (4.9) and using the chain rule, we get

$$\begin{aligned}\frac{p(z_1,\lambda)\partial_\lambda\log\left(rp(z_1,\lambda)\right) - p(z_2,\lambda)\partial_\lambda\log\left(rp(z_2,\lambda)\right)}{p(z_1,\lambda)-p(z_2,\lambda)}&\partial_{t_0}\lambda\\ = D(z_1)D(z_2)\partial_{t_0}\mathcal{F} &= -D(z_1)\log\left(rp(z_2,\lambda)\right)\\ = \frac{1}{2}\partial_\lambda\log\left(rp(z_2,\lambda)\right)\partial_\lambda&\log\left(rp(z_1,\lambda)\right)\frac{\partial_{t_0}\lambda}{\partial_\lambda\phi}.\end{aligned}$$

If $\partial_{t_0}\lambda \neq 0$, then we can rewrite this in the form

$$\begin{aligned}&(p(z_1,\lambda)\partial_\lambda\log(rp(z_1,\lambda)) - p(z_2,\lambda)\partial_\lambda\log(rp(z_2,\lambda)))\partial_\lambda\phi\\ &\quad= \tfrac{1}{2}\partial_\lambda\log(rp(z_2,\lambda))\partial_\lambda\log(rp(z_1,\lambda))(p(z_1,\lambda)-p(z_2,\lambda))\end{aligned}$$

which implies that

$$\frac{1}{\sigma(\lambda)} = \frac{1}{p(z,\lambda)} - \frac{2\partial_\lambda\phi(\lambda)}{p(z,\lambda)\partial_\lambda\left(\phi+\log p(z,\lambda)\right)} = -\frac{\partial_\lambda\phi - \partial_\lambda\log p(z,\lambda)}{p(z,\lambda)\left(\partial_\lambda\phi+\partial_\lambda\log p(z,\lambda)\right)} \tag{5.7}$$

is a constant independent of z. Rearranging, we obtain

$$\frac{\partial p(z,\lambda)}{\partial\lambda} = p(z,\lambda)\frac{\sigma(\lambda)+p(z,\lambda)}{\sigma(\lambda)-p(z,\lambda)}\frac{\partial\phi(\lambda)}{\partial\lambda},$$

which is the radial Löwner equation (2.2) for $p(z,\lambda)$.

Similarly, from the last equation of (4.9), we have

$$\tilde{D}(z)\lambda = \frac{1}{2}\partial_\lambda\log(r\tilde{p}(z,\lambda)^{-1})\frac{\partial_{t_0}\lambda}{\partial_\lambda\phi}. \tag{5.8}$$

Using this and applying ∂_{t_0} to the third equation of (4.9), we find that

$$\eta(\lambda) = \tilde{p}(z,\lambda) - \frac{2\tilde{p}(z,\lambda)\partial_\lambda\phi}{\partial_\lambda\phi - \partial_\lambda\log\tilde{p}(z,\lambda)} = -\tilde{p}(z,\lambda)\frac{\partial_\lambda\phi + \partial_\lambda\log\tilde{p}(z,\lambda)}{\partial_\lambda\phi - \partial_\lambda\log\tilde{p}(z,\lambda)}$$

is a constant independent of z. This gives us

$$\frac{\partial\tilde{p}(z,\lambda)}{\partial\lambda} = \tilde{p}(z,\lambda)\frac{\eta(\lambda)+\tilde{p}(z,\lambda)}{\eta(\lambda)-\tilde{p}(z,\lambda)}\frac{\partial\phi(\lambda)}{\partial\lambda},$$

which is the radial Löwner equation (2.1) for $\tilde{p}(z,\lambda)$.

Now, applying ∂_{t_0} to both sides of the second equation in (4.9), we have

$$\begin{aligned}\frac{-\partial_\lambda\tilde{p}(z_2,\lambda) + p(z_1,\lambda)^{-1}\tilde{p}(z_2,\lambda)\partial_\lambda p(z_1,\lambda)}{p(z_1,\lambda)-\tilde{p}(z_2,\lambda)}\partial_{t_0}\lambda &= -\tilde{D}(z_2)\log(rp(z_1,\lambda))\\ = -\frac{1}{2}\partial_\lambda\log(rp(z_1,\lambda))\partial_\lambda\log(r\tilde{p}(z_2,\lambda)^{-1})&\frac{\partial_{t_0}\lambda}{\partial_\lambda\phi}.\end{aligned}$$

Substituting the expressions for $\partial_\lambda p(z_1,\lambda)$ and $\partial_\lambda\tilde{p}(z_2,\lambda)$ obtained above, we get

$$-\frac{\eta(\lambda)+\tilde{p}(z_2,\lambda)}{\eta(\lambda)-\tilde{p}(z_2,\lambda)} + \frac{\sigma(\lambda)+p(z_1,\lambda)}{\sigma(\lambda)-p(z_1,\lambda)} = \frac{2\sigma(\lambda)(p(z_1,\lambda)-\tilde{p}(z_2,\lambda))}{(\eta(\lambda)-\tilde{p}(z_2,\lambda))(\sigma(\lambda)-p(z_1,\lambda))},$$

which after simplification implies that $\sigma(\lambda) = \eta(\lambda)$.

Finally using the Löwner equations for $p(z,\lambda)$ and $\tilde{p}(z,\lambda)$ and applying (3.5), equations (5.6) and (5.8) give

$$\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial\lambda}{\partial t_n} = -\frac{\sigma(\lambda)}{\sigma(\lambda)-p(z_1,\lambda)}\frac{\partial\lambda}{\partial t_0} = \sum_{n=1}^{\infty}\frac{\xi_n(\lambda)}{n}z^{-n}\frac{\partial\lambda}{\partial t_0}$$

$$\sum_{n=1}^{\infty}\frac{z^{n}}{n}\frac{\partial\lambda}{\partial t_{-n}} = -\frac{\tilde{p}(z_2,\lambda)}{\sigma(\lambda)-\tilde{p}(z_2,\lambda)}\frac{\partial\lambda}{\partial t_0} = \sum_{n=1}^{\infty}\frac{\xi_{-n}(\lambda)}{n}z^{n}\frac{\partial\lambda}{\partial t_0},$$

which imply

$$\frac{\partial\lambda}{\partial t_n} = \xi_n(\lambda)\frac{\partial\lambda}{\partial t_0}$$

for all $n\in\mathbb{Z}$. □

Remark 5.7. If we assume that $\mathcal{L}(w,\lambda(\boldsymbol{t}))$ maps the exterior disc $\mathbb{D}^*$ to a slit subdomain $\mathsf{B}\backslash\Gamma_\lambda$ of a domain B, then since $\mathcal{L}(\sigma(\lambda),\lambda)$ is the tip of Γ_λ, we have $|\sigma(\lambda)|=1$.

Remark 5.8. Since $\eta(\lambda)=\sigma(\lambda)$, it appears that $\mathcal{L}(w,\lambda)$ and $\tilde{\mathcal{L}}(w,\lambda)$ obey the same differential equation. However, one should note that $\mathcal{L}(w,\lambda)$ is a power series defined in a neighbourhood of infinity and $\tilde{\mathcal{L}}(w,\lambda)$ is a power series defined in a neighbourhood of the origin.

Similarly to the dKP case, the hydrodynamic-type equation (5.5) can be solved by the general hodograph method of Tsarev [32]:

Lemma 5.9. *Let $R(\lambda)$ be any function of λ. If $\lambda(\boldsymbol{t})$ is defined implicitly by the hodograph relation*

$$t_0+\sum_{n=1}^{\infty}\xi_n(\lambda)t_n+\sum_{n=1}^{\infty}\xi_{-n}(\lambda)t_{-n} = R(\lambda), \tag{5.9}$$

then $\lambda(\boldsymbol{t})$ satisfies (5.5).

Remark 5.10. If for some λ_0,

$$\tilde{\mathcal{L}}(w,\lambda_0) = \overline{\mathcal{L}(1/\bar{w},\lambda_0)}^{-1},$$

then by uniqueness of solutions to differential equations,

$$\tilde{\mathcal{L}}(w,\lambda) = \overline{\mathcal{L}(1/\bar{w},\lambda)}^{-1} \quad \text{for all } \lambda.$$

In this case,

$$w\Psi_n'(w,\lambda) = -\overline{\frac{1}{\bar{w}}\Phi_n'\left(\frac{1}{\bar{w}},\lambda\right)}.$$

If we also impose the condition $|\sigma(\lambda)|=1$, then

$$\xi_{-n}(\lambda) = -\overline{\xi_n(\lambda)}.$$

Therefore, if t_0 is a real-valued variable and for $n\neq 0$, t_n are complex-valued variables such that $t_{-n}=-\bar{t}_n$, then (5.9) defines $\lambda(\boldsymbol{t})$ as a real-valued function in a certain domain of $\boldsymbol{t}$.

As a converse to proposition 5.6, we have

Proposition 5.11. *Suppose*

$$G(w,\lambda)=\mathrm{e}^{\phi(\lambda)}w+\sum_{n=0}^{\infty}u_{n+1}(\lambda)w^{-n}$$

is a solution of the radial Löwner equation (2.2) and

$$F(w,\lambda)=\overline{G(1/\bar{w},\lambda)}^{-1}.$$

Let $\lambda(\boldsymbol{t})$ be a solution of equation (5.5), where $\boldsymbol{t}=\{t_n\}_{n\in\mathbb{Z}}, t_0$ is a real-valued variable and for $n\neq 0, t_n$ are complex variables such that $t_{-n}=-\bar{t}_n$. Under these conditions, $(\mathcal{L}(w;\boldsymbol{t}),\tilde{\mathcal{L}}(w;\boldsymbol{t}))$ defined by

$$\mathcal{L}(w;\boldsymbol{t})=G(w,\lambda(\boldsymbol{t}))\qquad\text{and}\qquad\tilde{\mathcal{L}}(w;\boldsymbol{t})=F(w,\lambda(\boldsymbol{t}))$$

is a solution of the dToda hierarchy (4.6) with $t_{-n}=-\bar{t}_n, n\geqslant 1$.

Proof. First, it is easy to verify that $F(w,\lambda)$ satisfies the radial Löwner equation (2.1) with $\eta(\lambda)=\sigma(\lambda)$.

Let $g(z,\lambda)$ and $f(z,\lambda)$ be the inverse functions of $G(w,\lambda)$ and $F(w,\lambda)$, respectively and let $b_{m,n}(\boldsymbol{t})=b_{m,n}(\lambda(\boldsymbol{t}))$, $m,n\in\mathbb{Z}$, be the Grunsky coefficients of $(f(z,\lambda),g(z,\lambda))$. Applying ∂_{t_k} to both sides of the first equation in (3.1) and using the Löwner equation for $g(z,\lambda)$, equation (5.5) and the first equation in (3.5), we have

$$\begin{aligned}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\partial b_{m,n}(\boldsymbol{t})}{\partial t_k}z_1^{-m}z_2^{-n}&=\frac{\partial_\lambda(\mathrm{e}^{\phi(\lambda)}g(z_1,\lambda))-\partial_\lambda(\mathrm{e}^{\phi(\lambda)}g(z_2,\lambda))}{\mathrm{e}^{\phi(\lambda)}(g(z_1,\lambda)-g(z_2,\lambda))}\partial_{t_k}\lambda\\&=\frac{2\sigma(\lambda)^2\partial_\lambda\phi\xi_k(\lambda)\partial_{t_0}\lambda}{(\sigma(\lambda)-g(z_1,\lambda))(\sigma(\lambda)-g(z_2,\lambda))}\\&=2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\xi_m(\lambda)\xi_n(\lambda)z_1^{-m}z_2^{-n}\xi_k(\lambda)\partial_\lambda\phi\partial_{t_0}\lambda.\end{aligned}$$

Similarly, applying ∂_{t_k} to both sides of equations in (3.3) and (3.2) and using the Löwner equations for $g(z,\lambda)$ and $f(z,\lambda)$, Lemma 5.9 and equations (3.5), we obtain

$$\begin{aligned}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\partial b_{m,-n}(\boldsymbol{t})}{\partial t_k}z_1^{-m}z_2^{n}&=2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\xi_m(\lambda)\xi_{-n}(\lambda)z_1^{-m}z_2^{n}\xi_k(\lambda)\partial_\lambda\phi\partial_{t_0}\lambda\\-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\partial b_{-m,-n}(\boldsymbol{t})}{\partial t_k}z_1^{m}z_2^{n}&=2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\xi_{-m}(\lambda)\xi_{-n}(\lambda)z_1^{m}z_2^{n}\xi_k(\lambda)\partial_\lambda\phi\partial_{t_0}\lambda\\-\sum_{n=1}^{\infty}\frac{\partial b_{n,0}(\boldsymbol{t})}{\partial t_k}z^{-n}&=-2\sum_{n=1}^{\infty}\frac{\xi_n(\lambda)}{n}z^{-n}\xi_k(\lambda)\partial_\lambda\phi\partial_{t_0}\lambda\\-\sum_{n=1}^{\infty}\frac{\partial b_{-n,0}(\boldsymbol{t})}{\partial t_k}z^{n}&=-2\sum_{n=1}^{\infty}\frac{\xi_{-n}(\lambda)}{n}z^{n}\xi_k(\lambda)\partial_\lambda\phi\partial_{t_0}\lambda.\end{aligned}$$

Comparing coefficients, we find that

$$2\xi_m(\lambda)\xi_n(\lambda)\xi_k(\lambda)\partial_\lambda\phi\partial_{t_0}\lambda=\begin{cases}-|mn|\partial_{t_k}b_{m,n}(\boldsymbol{t}), & \text{if}\quad |m|\neq 0, n\neq 0\\ |m|\partial_{t_k}b_{m,0}(\boldsymbol{t}), & \text{if}\quad m\neq 0, n=0\\ 2\partial_{t_k}\phi(\lambda(\boldsymbol{t})), & \text{if}\quad m=n=0.\end{cases}$$

Since the left-hand side is completely symmetric in m, n, k, we conclude that there exists a function $\mathcal{F}(\boldsymbol{t})$ such that

$$\frac{\partial^2 \mathcal{F}(\boldsymbol{t})}{\partial t_m \partial t_n} = \begin{cases} -|mn| b_{m,n}(\boldsymbol{t}), & \text{if} \quad m \neq 0, n \neq 0 \\ |m| b_{m,0}(\boldsymbol{t}), & \text{if} \quad m \neq 0, n = 0 \\ 2\phi(\lambda(\boldsymbol{t})), & \text{if} \quad m = n = 0. \end{cases}$$

Since $\phi(\lambda) = -b_{0,0}(\lambda)$, the assertion follows from proposition 4.2. □

Remark 5.12. In [19], Mañas has considered the reduction of rth dispersionless Dym (dDym) hierarchy. When $r = 1$, the dDym hierarchy is gauge equivalent to 'half' of the dToda hierarchy (4.6), where only the power series $\mathcal{L}(w, \boldsymbol{t})$ depending on $\boldsymbol{t} = (t_0, t_1, t_2, \ldots)$[6] is present. Mañas proved a result similar to proposition 5.11 by a completely different method.

Recently, Takasaki and Takebe [29] also showed that a solution to the radial Löwner equation together with the hydrodynamic-type equation (5.5) for $n \geqslant 1$ generates a solution to the first series of the Lax equations of the dToda hierarchy (the first equation in (4.6). In fact, the results of [29] are easily recovered from [19] by just adding linear and constant terms in p to reduction condition (30) of [19].

6. Large *N* eigenvalue integrals and reductions of dispersionless hierarchies

Here we explain the relations between dispersionless integrable hierarchies, their reductions and conformal maps from the perspective of the model of normal random matrices [6, 7]. Actually, we need only the eigenvalue integral for that model, which represents the partition function of the 2D Coulomb gas in external field. In this section, the exposition is on the physical level of rigour.

6.1. Large N integrals and dToda hierarchy

A convenient starting point is the N-fold integral

$$\tau_N = \frac{1}{N!} \int_{\mathbb{C}^N} \prod_{i<j}^{N} |z_i - z_j|^2 \prod_{k=1}^{N} \exp\left(\frac{1}{\hbar} \sum_{n=1}^{\infty} \left(t_n z_k^n + \bar{t}_n \bar{z}_k^n \right) \right) \mathrm{d}\mu(z_k, \bar{z}_k), \tag{6.1}$$

where $\mathrm{d}\mu$ is some integration measure in the complex plane, and $\hbar$ is a parameter. For $\mathrm{d}\mu = \exp\left(\frac{1}{\hbar} W(z, \bar{z})\right) \mathrm{d}^2 z$, τ_N is the partition function of the model of normal random matrices with the potential W written as an integral over eigenvalues. Clearly, τ_N (6.1) has the meaning of the partition function for a system of N 2D Coulomb charges in an external potential. The basic fact about the integral (6.1) is as follows.

- For any measure $\mathrm{d}\mu$ (including singular measures supported on sets of dimension less than 2), τ_N, as a function of $t_0 = N\hbar$, $\{t_n\}$, $\{-\bar{t}_n\}$, is a τ-function of the (dispersionful) 2D Toda hierarchy with $t_{-n} = -\bar{t}_n$.

In a slightly different form, this statement first appeared in [14], see also [7].

In the large N limit ($N \to \infty, \hbar \to 0, t_0 = \hbar N$ finite) τ_N generates the 'free energy' $\mathcal{F}$ for the dToda hierarchy via

$$\mathcal{F}(\boldsymbol{t}) = \lim_{N \to \infty} (\hbar^2 \log \tau_N).$$

where $\boldsymbol{t} = \{\ldots, -\bar{t}_2, -\bar{t}_1, t_0, t_1, t_2, \ldots\}$. It obeys the dispersionless Hirota equations (4.9). Second order derivatives of $\mathcal{F}$ enjoy a nice geometric interpretation through conformal maps.

[6] The variable t_0 should be identified with x in [19].

In short, this goes as follows. As $N \to \infty$, the integral in (6.1) is determined by the most favourable configuration of z_i's, i.e., the one at which the integrand has a maximum (which becomes very sharp in the large N limit). In other words, the free energy $\mathcal{F}$ is essentially the electrostatic energy of the equilibrium configuration of the 2D Coulomb charges in the external potential. Exploiting further the electrostatic analogy, one can see that in the equilibrium the 'charges' densely fill a bounded domain D in the complex plane. For simplicity, we assume that D is connected. In the matrix model interpretation, this domain is called the support of eigenvalues. Clearly, it is a compact subset of the support of the measure $\mathrm{d}\mu$.

Let $p(z)$ be the conformal mapping function from the *exterior* of the domain D onto the exterior of the unit circle normalized as $p(z) = z/r + O(1)$ at large $|z|$ with a real r called the exterior conformal radius of the domain D. In [21, 23, 34, 36] it was shown that the function $p(z)$ can be expressed through $\mathcal{F}$ in the following different but equivalent ways:

$$rp(z) = z\,\mathrm{e}^{-D(z)\partial_{t_0}\mathcal{F}}, \qquad rp(z) = z - a - D(z)\partial_{t_1}\mathcal{F}, \qquad rp^{-1}(z) = D(z)\partial_{\bar{t}_1}\mathcal{F} \tag{6.2}$$

where

$$2\log r = \frac{\partial^2\mathcal{F}}{\partial t_0^2}, \qquad a = \frac{\partial^2\mathcal{F}}{\partial t_0 \partial t_1}$$

and the operator $D(z)$ is defined in (4.8). Moreover, the free energy obeys the Hirota equations (4.9) with $\tilde{p}(z) = 1/\overline{p(1/\bar{z})}$. It is convenient to rewrite them in terms of the function

$$\bar{p}(z) = \overline{p(\bar{z})} = r^{-1} z\,\mathrm{e}^{-\bar{D}(z)\partial_{t_0}\mathcal{F}} \qquad \text{where} \qquad \bar{D}(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\frac{\partial}{\partial \bar{t}_n}$$

We have

$$\begin{aligned} D(z_1)D(z_2)\mathcal{F} &= \log\frac{rp(z_1) - rp(z_2)}{z_1 - z_2}, \\ \bar{D}(z_1)\bar{D}(z_2)\mathcal{F} &= \log\frac{r\bar{p}(z_1) - r\bar{p}(z_2)}{z_1 - z_2} \\ -D(z_1)\bar{D}(z_2)\mathcal{F} &= \log\left(1 - \frac{1}{p(z_1)\bar{p}(z_2)}\right). \end{aligned} \tag{6.3}$$

We emphasize that all these relations hold true for *any* measure $\mathrm{d}\mu$ in (6.1) provided the equilibrium configuration of z_i's at $N \to \infty$ is well defined.

In particular, one may consider the measure supported on a curve Γ, then the integral (6.1) becomes one dimensional (along Γ) in each variable:

$$\tau_N = \frac{1}{N!}\int_{\Gamma^N} \prod_{i<j}^{N} |z_i - z_j|^2 \prod_{k=1}^{N} \exp\left(\frac{1}{\hbar}\sum_{n\geqslant 1}\left(t_n z_k^n + \bar{t}_n \bar{z}_k^n\right)\right) |\mathrm{d}z_k|. \tag{6.4}$$

In the large N limit, the support of eigenvalues, D, is then an arc of the curve Γ (or several disconnected arcs, but we do not consider this case in the present paper). The function $p(z)$ maps the slit domain $\mathbb{C}\backslash\mathsf{D}$ onto the exterior of the unit circle. (See figure 4.)

For the dToda hierarchy, the choice of the measure supported on a curve means *a reduction*. A familiar example is the dToda chain, where one may take Γ to be either real or imaginary axis. Consider a general (continuous) curve Γ infinite in both directions. It is clear that $p(z)$ and the Lax functions (the functions inverse to $p(z)$ and $1/\bar{p}(1/z)$) depend on the times through two parameters only. One can set them to be, for example, the positions of the two ends of the arc D on the curve Γ (although this is not a best choice). Reductions with only one independent parameter are also possible. The simplest possibility of obtaining them from

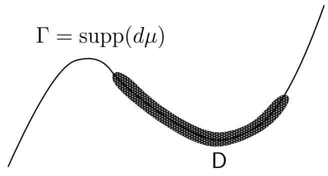

Figure 4. The supports of the measure and of eigenvalues.

the large N integral (6.1) is to take the measure dμ supported on a half-infinite curve, starting at 0 (for example) and going to infinity, and such that the arc D starts from 0 as the times independently vary in some open set. A more general class of examples can be obtained in a similar way if one restricts the measure to be supported on the boundary of some domain B and on a curve Γ starting on the boundary and coming to infinity. Under certain conditions, only the second edge of the arc moves, so that we are left with one real parameter $\lambda = \lambda(\boldsymbol{t})$. We have shown in section 5.2 that in this case, the corresponding conformal map $p(z)$ satisfies the radial Löwner equation.

6.2. The reflection symmetry case: the chordal Löwner equation from reductions of the dToda hierarchy

The case when the measure dμ has a symmetry is of special interest. In order to respect the symmetry, one may need to restrict the flows of the full dToda hierarchy to a submanifold of the space with coordinates $t_n, \bar{t}_n$. Reductions of the so-obtained sub-hierarchy may lead to interesting classes of conformal maps. Below we study the case of reflection symmetry.

Suppose the measure is symmetric under the reflection in the real axis, i.e., d$\mu(z, \bar{z}) =$ d$\mu(\bar{z}, z)$. We will show that this case allows one to relate conformal maps described by the chordal Löwner equation (2.3) to the conformal maps generated by solutions of the dToda hierarchy. The relation is not immediately obvious.

The symmetry of the measure does not yet imply the symmetry of the support of eigenvalues since the latter depends also on the function $\sum_n (t_n z^n + \bar{t}_n \bar{z}^n)$ which does not enjoy the reflection symmetry unless all t_k's are real. If they are, then the domain D is invariant under the reflection (complex conjugation). In this case the conformal map has real coefficients: $\bar{p}(z) = p(z)$, i.e., $\partial_{t_0}\partial_{t_n}\mathcal{F} = \partial_{t_0}\partial_{\bar{t}_n}\mathcal{F}$. Equations (6.3) then imply that in the real section of the times manifold ($t_n = \bar{t}_n$) it holds $\partial_{t_m}\partial_{t_n}\mathcal{F} = \partial_{\bar{t}_m}\partial_{\bar{t}_n}\mathcal{F}$ and $\partial_{t_m}\partial_{\bar{t}_n}\mathcal{F} = \partial_{\bar{t}_m}\partial_{t_n}\mathcal{F}$ for any m, n. Therefore, the dToda hierarchy restricted to this real section describes conformal maps of symmetric domains. Unfortunately, we do not know whether the restricted flows form a reasonable hierarchy.

Deformations which destroy the reflection symmetry of D can be parametrized by the times $s_n = \frac{1}{\sqrt{2}}(t_n - \bar{t}_n)$ (we define them to be purely imaginary for later convenience). We are going to show that infinitesimal deformations of this kind are described by the dKP hierarchy in the times $\boldsymbol{s} = \{s_n\}$. More precisely, the Hirota equation of the form (4.4) generating the dKP hierarchy in the times $\boldsymbol{s}$ holds true at the point $\boldsymbol{s} = 0$, and for s_n of order ϵ the corrections to this equation are of order ϵ^2.

Let us take the sum of the three equations (6.3) and the fourth equation obtained from the third one in (6.3) by the interchange $z_1 \leftrightarrow z_2$. After exponentiating, we get, using the symmetry $\bar{p}(z) = p(z)$:

$$(z_1 - z_2)\exp\left(\tfrac{1}{2}(D(z_1) - \bar{D}(z_1))(D(z_2) - \bar{D}(z_2))\mathcal{F}\right)$$
$$= r(p(z_1) - p(z_2) + p^{-1}(z_1) - p^{-1}(z_2)).$$

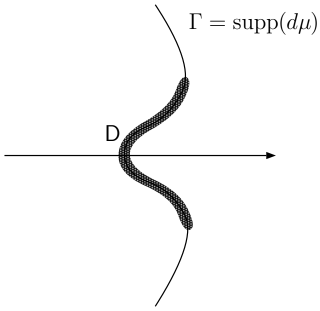

Figure 5. Reflection symmetry case; a slit domain.

Summing three equations of this type written for all pairs of the points z_1, z_2, z_3, we obtain

$$(z_1 - z_2)\exp(D^{\mathrm{KP}}(z_1)D^{\mathrm{KP}}(z_2)\mathcal{F}) + (z_2 - z_3)\exp(D^{\mathrm{KP}}(z_2)D^{\mathrm{KP}}(z_3)\mathcal{F}) + (z_3 - z_1)\exp(D^{\mathrm{KP}}(z_3)D^{\mathrm{KP}}(z_1)\mathcal{F}) = 0, \tag{6.5}$$

where

$$D^{\mathrm{KP}}(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{s_n}, \qquad \partial_{s_n} = \frac{1}{\sqrt{2}}(\partial_{t_n} - \partial_{\bar{t}_n}).$$

Setting

$$k(z) = z - D^{\mathrm{KP}}(z)\partial_{s_1}\mathcal{F} \tag{6.6}$$

and letting $z_3 \to \infty$ in (6.5), we get

$$D^{\mathrm{KP}}(z_1)D^{\mathrm{KP}}(z_2)\mathcal{F} = \log\frac{k(z_1) - k(z_2)}{z_1 - z_2}, \tag{6.7}$$

which is the Hirota equation for the dKP hierarchy (4.4). It is important to note that equations (6.5), (6.7) *hold at the point* $\boldsymbol{s} = 0$ *only* and thus the evolution in s_n that they define is not consistent with that defined by the dToda hierarchy for the same function $\mathcal{F}$. However, for small s_n the two evolutions agree up to higher order terms. More precisely, let all the s_n's be of order $\epsilon \to 0$, then the coefficients of $p(z)$ acquire imaginary parts of the same order ϵ (or higher) while their real parts are not changed up to this order. Summing the equations (6.3) as before but without the assumption that $\bar{p}(z) = p(z)$, one can see that the terms of order ϵ in the rhs cancel. Therefore, the lhs of (6.5) is of order ϵ^2. This means that the equation obtained from (6.5) by applying any derivative ∂_{s_n} at $s_n = 0$ is still valid. The same is true for equation (6.7). As we have seen, this is enough to derive the chordal Löwner equation from the dKP hierarchy (with a reduction imposed). Therefore, conformal maps of slit domains *symmetric under reflection in the real axis* satisfy the chordal Löwner equation (2.3), as they should. Note that symmetric slits have only one parameter (the curve Γ is fixed). (See figure 5.)

It remains to verify that the function $k(z)$ defined by (6.6) is indeed the conformal map with the required properties[7]:

(i) Normalization $k(z) = z + O(1/z)$ as $z \to \infty$;

[7] In the Löwner equation literature, the map $k(z)$ is usually regarded as a map from the upper half plane with a slit onto the upper half plane. The Schwarz symmetry principle allows one to analytically continue the map to the lower half plane, with the result being characterized by these properties.

(ii) real coefficients;
(iii) the image is the complement of a segment of the real axis.

The first two are obvious from the representation (6.6). The third one follows from the identity

$$k(z) = r(p(z) + p^{-1}(z)) + a \tag{6.8}$$

which is a direct consequence of the second and third equations in (6.2). Indeed, the function $r(w + w^{-1}) + a$ maps the exterior of the unit circle in the w-plane, which is the image of the map $p(z)$, onto the exterior of a segment of the real line. The function $k(z)$ is thus the composition of the two maps.

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Appendix A. Examples

A.1. Chordal Löwner equation and dKP hierarchy

A.1.1. Example 1. We consider the chordal Löwner equation (2.3) with $a_1 = -\lambda$, $U(\lambda) = U \in \mathbb{R}$ a constant and initial condition $H(w, 0) = w$. It is easy to find that its solution is given by

$$\begin{aligned} H(w,\lambda) &= U + w\sqrt{1 - 2Uw^{-1} + (U^2 - 2\lambda)w^{-2}} = w - \frac{\lambda}{w} - \frac{\lambda U}{w^2} + \cdots \\ h(z,\lambda) &= U + z\sqrt{1 - 2Uz^{-1} + (U^2 + 2\lambda)z^{-2}} = z + \frac{\lambda}{z} + \frac{\lambda U}{z^2} + \cdots \end{aligned} \tag{A.1}$$

$H(w, \lambda)$ maps the upper half plane $\mathbb{H}$ conformally onto $\mathbb{H}\backslash\Gamma_\lambda$, where Γ_λ is the line segment

$$\Gamma_\lambda = \{U + \mathrm{i}\alpha : \alpha \in [0, \sqrt{2\lambda}]\}.$$

Let $\Phi_n(w, \lambda), n \geqslant 1$, be the Faber polynomials of $h(z, \lambda)$. A straightforward computation gives

$$\Phi_1(w,\lambda) = w, \qquad \Phi_2(w,\lambda) = w^2 - 2\lambda, \qquad \Phi_3(w,\lambda) = w^3 - 3\lambda w - 3\lambda U.$$

Therefore the functions $\chi_n(\lambda), n = 1, 2, 3$, in (5.1) are given by

$$\chi_1(\lambda) = 1, \qquad \chi_2(\lambda) = 2U, \qquad \chi_3(\lambda) = 3U^2 - 3\lambda.$$

When $t_n = 0$ for all $n \geqslant 4$, the hodograph relation (5.4) with $R(\lambda) = 0$ reads as

$$x + t_1 + 2Ut_2 + (3U^2 - 3\lambda)t_3 = 0$$

which for $\{x + t_1 + 2Ut_2 + 3U^2t_3 \geqslant 0 \text{ and } t_3 > 0\}$ or $\{x + t_1 + 2Ut_2 + 3U^2t_3 \leqslant 0 \text{ and } t_3 < 0\}$ gives

$$\lambda = \frac{x + t_1 + 2Ut_2 + 3U^2t_3}{3t_3} \geqslant 0.$$

Substituting into the first equation of (A.1), we find that the power series

$$\mathcal{L}(w; \boldsymbol{t})|_{t_n=0(n\geqslant 4)} = U + w\sqrt{1 - 2Uw^{-1} - \left(U^2 + \frac{2(x + t_1 + 2Ut_2)}{3t_3}\right)w^{-2}}$$

is a solution of the dKP hierarchy (4.2).

A.1.2. Example 2. Let

$$H(w,\lambda) = w + \frac{\lambda^2}{w-2\lambda} = w + \lambda^2 w^{-1} + 2\lambda^3 w^{-2} + 4\lambda^4 w^{-3} + \cdots. \tag{A.2}$$

For $\lambda > 0$, $H(w,\lambda)$ maps the upper half plane conformally onto

$$\mathbb{C}\backslash(\{x : x \in (-\infty, 0]\} \cup \{x : x \in [4\lambda, \infty)\}).$$

It is easy to check that $H(w,\lambda)$ satisfies the chordal Löwner equation (2.3) with $a_1 = \lambda^2$ and $U(\lambda) = 3\lambda$. Let $\Phi_n(w,\lambda)$, $n \geqslant 1$ be the Faber polynomials of $h(z,\lambda)$, the inverse function of $H(w,\lambda)$, then

$$\Phi_1(w,\lambda) = w, \qquad \Phi_2(w,\lambda) = w^2 + 2\lambda^2, \qquad \Phi_3(w,\lambda) = w^3 + 3\lambda^2 w + 6\lambda^3.$$

The functions $\chi_n(\lambda)$, $n = 1, 2, 3$, defined in (5.1) are given by

$$\chi_1(\lambda) = 1, \qquad \chi_2(\lambda) = 6\lambda, \qquad \chi_3(\lambda) = 30\lambda^2.$$

When $t_n = 0$ for all $n \geqslant 3$, the hodograph relation (5.4) with $R(\lambda) = 0$ reads as

$$x + t_1 + 6t_2\lambda = 0,$$

which for $\{x + t_1 \geqslant 0$ and $t_2 < 0\}$ or $\{x + t_1 \leqslant 0$ and $t_2 > 0\}$ gives

$$\lambda = -\frac{x+t_1}{6t_2} \geqslant 0.$$

Substituting into (A.2), we find that the power series

$$\mathcal{L}(w,\boldsymbol{t})|_{t_n=0(n\geqslant 3)} = w + \frac{(x+t_1)^2}{12t_2}\frac{1}{3t_2 w + (x+t_1)}$$

is a solution of the dKP hierarchy (4.2). When $t_n = 0$ for all $n \geqslant 4$, the hodograph relation (5.4) with $R(\lambda) = 0$ reads as

$$x + t_1 + 6t_2\lambda + 30t_3\lambda^2 = 0,$$

which can be solved for $\lambda(\boldsymbol{t})|_{t_n=0(n\geqslant 4)}$ in a certain domain of $\boldsymbol{t}$.

A.2. Radial Löwner equation and dToda hierarchy

A.2.1. Example 1. We consider the radial Löwner equation (2.2) with $\phi(\lambda) = \lambda$, $\sigma(\lambda) = \sigma \in S^1$ a constant and initial condition $G(w, 0) = w$. It is easy to check that its solution is given by[8]

$$\begin{aligned} G(w,\lambda) &= -\sigma + \frac{e^{\lambda} w}{2}\left(\left(1+\frac{\sigma}{w}\right)^2 + \left(1+\frac{\sigma}{w}\right)\sqrt{1+\frac{2\sigma(1-2e^{-\lambda})}{w}+\frac{\sigma^2}{w^2}}\right) \\ &= e^{\lambda} w + 2\sigma(e^{\lambda}-1) + \frac{\sigma^2(e^{\lambda}-e^{-\lambda})}{w} + \frac{2\sigma^3 e^{-\lambda}(1-e^{-\lambda})}{w^2} + \cdots \\ g(z,\lambda) &= -\sigma + \frac{e^{-\lambda} z}{2}\left(\left(1+\frac{\sigma}{z}\right)^2 + \left(1+\frac{\sigma}{z}\right)\sqrt{1+\frac{2\sigma(1-2e^{\lambda})}{z}+\frac{\sigma^2}{z^2}}\right). \end{aligned} \tag{A.3}$$

$G(w,\lambda)$ maps the exterior disc $\mathbb{D}^*$ conformally onto $\mathbb{D}^*\backslash\Gamma_\lambda$, where Γ_λ is the line segment

$$\Gamma_\lambda = \{\alpha\sigma : \alpha \in [1, 2e^{\lambda} - 1 + 2\sqrt{e^{2\lambda} - e^{\lambda}}]\}.$$

[8] In fact, up to a re-parametrization of λ, this example is the same as that given in [29].

The maps

$$F(w,\lambda) = -\sigma + \frac{\mathrm{e}^{\lambda}}{2w}\left((w+\sigma)^2 - \sigma(w+\sigma)\sqrt{1+2\sigma^{-1}(1-2\mathrm{e}^{-\lambda})w+\sigma^{-2}w^2}\right)$$
$$f(z,\lambda) = -\sigma + \frac{\mathrm{e}^{-\lambda}}{2z}\left((z+\sigma)^2 - \sigma(z+\sigma)\sqrt{1+2\sigma^{-1}(1-2\mathrm{e}^{\lambda})z+\sigma^{-2}z^2}\right) \tag{A.4}$$

satisfy the radial Löwner equations (2.1) with $\eta(\lambda) = \sigma$ and initial condition $F(w,0) = w$. It is easy to verify directly that

$$F(w,\lambda) = \overline{G(1/\bar{w},\lambda)}^{-1}. \tag{A.5}$$

Therefore, $F(w,\lambda)$ maps the unit disc $\mathbb{D}$ conformally onto $\mathbb{D}\backslash\tilde{\Gamma}_\lambda$, where $\tilde{\Gamma}_\lambda$ is the line segment

$$\tilde{\Gamma}_\lambda = \{\alpha\sigma : \alpha \in [2\mathrm{e}^{\lambda} - 1 - 2\sqrt{\mathrm{e}^{2\lambda}-\mathrm{e}^{\lambda}}, 1]\}.$$

Let $J_\lambda = g(\Gamma_\lambda,\lambda)$. In fact, for fixed λ, $F(w,\lambda)$ is the analytic continuation of $G(w,\lambda)$ to the set $\mathbb{D}\cup(S^1\backslash J_\lambda)$.

Let $\Phi_n(w,\lambda)$, $\Psi_n(w,\lambda)$, $n \geqslant 1$, be the Faber polynomials of $g_\lambda(z)$ and $f_\lambda(z)$, respectively. Then

$$\Phi_1(w,\lambda) = \mathrm{e}^{\lambda}w + 2\sigma(\mathrm{e}^{\lambda}-1), \qquad \Psi_1(w,\lambda) = \mathrm{e}^{\lambda}w^{-1} + 2\sigma^{-1}(\mathrm{e}^{\lambda}-1),$$
$$\Phi_2(w,\lambda) = \mathrm{e}^{2\lambda}w^2 + 4\sigma\,\mathrm{e}^{\lambda}(\mathrm{e}^{\lambda}-1)w + 2\sigma^2(\mathrm{e}^{\lambda}-1)(2\mathrm{e}^{\lambda}-1+\mathrm{e}^{-\lambda}),$$
$$\Psi_2(w,\lambda) = \mathrm{e}^{2\lambda}w^{-2} + 4\sigma^{-1}\,\mathrm{e}^{\lambda}(\mathrm{e}^{\lambda}-1)w^{-1} + 2\sigma^{-2}(\mathrm{e}^{\lambda}-1)(2\mathrm{e}^{\lambda}-1+\mathrm{e}^{-\lambda}).$$

Therefore, the functions $\xi_n(\lambda)$, $n = \pm1, \pm2$ in (5.5) are given by

$$\xi_1(\lambda) = \sigma\,\mathrm{e}^{\lambda}, \qquad \xi_{-1}(\lambda) = -\sigma^{-1}\,\mathrm{e}^{\lambda}$$
$$\xi_2(\lambda) = 2\sigma^2\,\mathrm{e}^{\lambda}(3\mathrm{e}^{\lambda}-2), \qquad \xi_{-2}(\lambda) = -2\sigma^{-2}\mathrm{e}^{\lambda}(3\mathrm{e}^{\lambda}-2).$$

When $t_{-1} = -\bar{t}_1$ and $t_n = 0$ for $|n| \geqslant 2$, the hodograph relation (5.9) with $R(\lambda) = 0$ reads as

$$t_0 + t_1\sigma\,\mathrm{e}^{\lambda} + \bar{t}_1\bar{\sigma}\,\mathrm{e}^{\lambda} = 0. \tag{A.6}$$

When $t_0 \geqslant -2\,\mathrm{Re}(t_1\sigma) > 0$ or $-t_0 \geqslant 2\,\mathrm{Re}(t_1\sigma) > 0$, this gives

$$\lambda = \log\left(-\frac{t_0}{t_1\sigma + \bar{t}_1\bar{\sigma}}\right) \geqslant 0.$$

Substituting into the first equation in (A.3) and the first equation in (A.4), we find that the power series

$$\mathcal{L}(w;\boldsymbol{t})|_{t_n=0(|n|\geqslant2)}$$
$$= -\sigma - \frac{t_0 w}{4\,\mathrm{Re}(t_1\sigma)}\left(\left(1+\frac{\sigma}{w}\right)^2 + \left(1+\frac{\sigma}{w}\right)\sqrt{1+\frac{2\sigma(t_0+4\,\mathrm{Re}(t_1\sigma))}{wt_0}+\frac{\sigma^2}{w^2}}\right)$$

$$\tilde{\mathcal{L}}(w;\boldsymbol{t})|_{t_n=0(|n|\geqslant2)}$$
$$= -\sigma - \frac{t_0}{4\,\mathrm{Re}(t_1\sigma)w}\left((\sigma+w)^2 - \sigma\,(\sigma+w)\sqrt{1+\frac{2\bar{\sigma}(t_0+4\,\mathrm{Re}(t_1\sigma))w}{t_0}+\bar{\sigma}^2w^2}\right)$$

satisfy the dToda hierarchy (4.6) with $t_{-n} = -\bar{t}_n$ for $n \geqslant 1$.

When $t_{-n} = -\bar{t}_n$ $\forall n \geqslant 1$ and $t_n = 0$ for $|n| \geqslant 3$, the hodograph relation (5.9) with $R(\lambda) = 0$ reads as

$$t_0 + t_1\sigma\,\mathrm{e}^{\lambda} + \bar{t}_1\bar{\sigma}\,\mathrm{e}^{\lambda} + 2t_2\sigma^2\,\mathrm{e}^{\lambda}(3\mathrm{e}^{\lambda}-2) + 2\bar{t}_2\bar{\sigma}^2\,\mathrm{e}^{\lambda}(3\mathrm{e}^{\lambda}-2) = 0,$$

which can be solved for $\lambda(\boldsymbol{t})|_{t_n=0(|n|\geqslant3)}$ in a certain domain of $\boldsymbol{t}$.

Remark A.1. If we do not impose the condition $|\sigma| = 1$ in the example above, then the maps $G_\lambda(w)$ and $F_\lambda(w)$ still solve the Löwner equations (2.2), (2.1) with initial conditions $G_0(w) = w$ and $F_0(w) = w$, respectively. In this case, $G_\lambda(w)$ maps the disc $\{|w| \geqslant |\sigma|\}$ conformally onto $\{|z| \geqslant |\sigma|\}\backslash\{\alpha\sigma : \alpha \in [1, 2e^\lambda - 1 + 2\sqrt{e^{2\lambda} - e^\lambda}]\}$ and $F_\lambda(w)$ maps the disc $\{|w| \leqslant |\sigma|\}$ conformally onto $\{|z| \leqslant |\sigma|\}\backslash\{\alpha\sigma : \alpha \in [2e^\lambda - 1 - 2\sqrt{e^{2\lambda} - e^\lambda}, 1]\}$. However, (A.5) no longer holds. Nevertheless, one can still show as in proposition 5.11 that with $\lambda(\boldsymbol{t})$ satisfying (5.5), the conclusion of proposition 5.11 still holds.

A.2.2. Example 2. Given a point $\sigma \in S^1$, let

$$G(w, \lambda) = e^\lambda(w + 2\sigma + \sigma^2 w^{-1}) = \frac{e^\lambda(w + \sigma)^2}{w}.$$

It is easy to verify directly that $G(w, \lambda)$ satisfies the radial Löwner equation (2.2) with $\sigma(\lambda) = \sigma$ and $\phi(\lambda) = \lambda$. Therefore, $G(w, \lambda)$ here satisfies the same equation as the $G(w, \lambda)$ in example 1 but with a different initial condition. It maps the exterior disc conformally onto

$$\hat{\mathbb{C}}\backslash\{\sigma\alpha : \alpha \in [0, 4e^\lambda]\}.$$

The function

$$F(w, \lambda) = \overline{G(1/\bar{w}, \lambda)}^{-1} = \frac{e^{-\lambda} w}{(1 + \bar{\sigma} w)^2} = e^{-\lambda}(w + 2\bar{\sigma} w^2 + 3\bar{\sigma}^2 w^3 + \cdots)$$

maps $\mathbb{D}$ conformally onto

$$\mathbb{C}\backslash\{\sigma\alpha : \alpha \in [e^{-\lambda}/4, \infty)\}.$$

For $n \geqslant 1$, let $\Phi_n(w, \lambda)$ and $\Psi_n(w, \lambda)$ be the Faber polynomials of the inverse functions $g(z, \lambda)$, $f(z, \lambda)$ of $G(w, \lambda)$, $F(w, \lambda)$, respectively. Then

$$\Phi_1(w, \lambda) = e^\lambda(w + 2\sigma), \qquad \Psi_1(w, \lambda) = e^\lambda(w^{-1} + 2\bar{\sigma}),$$
$$\Phi_2(w, \lambda) = e^{2\lambda}(w^2 + 4\sigma w + 6\sigma^2), \qquad \Psi_2(w, \lambda) = e^{2\lambda}(w^2 + 4\bar{\sigma} w + 6\bar{\sigma}^2).$$

Therefore, the functions $\xi_n(\lambda)$, $n = \pm1, \pm2$ in (5.5) are given by

$$\xi_1(\lambda) = e^\lambda\sigma, \qquad \xi_{-1}(\lambda) = -e^\lambda\bar{\sigma},$$
$$\xi_2(\lambda) = 6e^{2\lambda}\sigma^2, \qquad \xi_{-2}(\lambda) = -6e^{2\lambda}\bar{\sigma}^2.$$

When $t_{-1} = -\bar{t}_1$ and $t_n = 0$ for $|n| \geqslant 2$, the hodograph relation (5.9) with $R(\lambda) = 0$ is the same as equation (A.6). Therefore, when $t_0 \geqslant -2\,\mathrm{Re}(t_1\sigma) > 0$ or $-t_0 \geqslant 2\,\mathrm{Re}(t_1\sigma) > 0$, we find that the power series

$$\mathcal{L}(w; \boldsymbol{t})|_{t_n=0(|n|\geqslant 2)} = -\frac{t_0}{2\,\mathrm{Re}(t_1\sigma)}(w + 2\sigma + \sigma^2 w^{-1})$$
$$\tilde{\mathcal{L}}(w; \boldsymbol{t})|_{t_n=0(|n|\geqslant 2)} = -\frac{2\,\mathrm{Re}(t_1\sigma)}{t_0}\frac{w}{(1 + \bar{\sigma} w)^2}$$

satisfies the dToda hierarchy (4.6) with $t_{-n} = -\bar{t}_n$ for $n \geqslant 1$. When $t_{-n} = -\bar{t}_n$ $\forall n \geqslant 1$ and $t_n = 0$ for $|n| \geqslant 3$, the hodograph relation (5.9) with $R(\lambda) = 0$ reads as

$$t_0 + t_1\sigma\, e^\lambda + \bar{t}_1\bar{\sigma}\, e^\lambda + 6t_2\, e^{2\lambda}\sigma^2 + 6\bar{t}_2\, e^{2\lambda}\bar{\sigma}^2 = 0,$$

which gives

$$\lambda(\boldsymbol{t})|_{t_n=0(|n|\geqslant 3)} = \log\left(\frac{-\mathrm{Re}(t_1\sigma) + \sqrt{[\mathrm{Re}(t_1\sigma)]^2 - 12t_0\,\mathrm{Re}(t_2\sigma^2)}}{12\,\mathrm{Re}(t_2\sigma^2)}\right)$$

in a certain domain of $\boldsymbol{t}$.

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